DAMPING OF STEADY-STATE WAVES IN SYSTEMS DESCRIBED BY A NONLINEAR KLEIN–GORDON EQUATION

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The damping of a nonsinusoidal wave in systems described by a Klein–Gordon equation is investigated by the method of averaging. An explicit solution is given for an initial-value problem. It is shown that in certain cases the prolonged existence of a steady-state wave is impossible. Dissipation can lead to the damping out of the wave. The characteristic features of the boundary-value problem are discussed. Formulas are obtained describing the damping of single pulses (solitons).

1. A nonlinear wave equation of the form

$$\frac{\partial^2\varphi}{\partial t^2} - c^2\frac{\partial^2\varphi}{\partial x^2} + \frac{c^2}{\Lambda^2}\sin\varphi = 0 \tag{1.1}$$

known as the sine-Gordon equation, is widely used in various physics problems. It describes the motion of domain boundaries in ferromagnets [1] and the distribution of dislocations in a crystal lattice [2], the propagation of an ultrashort light pulse in a nonlinear active medium [3], and the electromagnetic field in Josephson junctions [4]. Equation (1.1) is used in certain models of the theory of fields [5] and in rigid-body mechanics [6]. Solutions of Eq. (1.1) have been investigated in differential geometry [7], and exact solutions describing the interaction of two waves were obtained for the first time in nonlinear theory. Multiwave (N-soliton) solutions have been found also [8, 9].

The effect of dissipation on nonlinear waves has been studied for the slightly nonlinear case [4, 10, 11]. Since the solution of the problem of the propagation of a wave taking account of absorption is not known in general form, a class of quasistationary waves characterized by amplitude and frequency only can be considered for an arbitrary value of the nonlinearity and analyzed by using averaging methods. Such waves have been discussed in the literature in the absence of absorption, and their stability with respect to perturbations of the envelopes has been studied [6, 12-14]. In the present paper we solve the problem of the damping of a wave with a steady-state shape at zero time.

Steady-state waves are distinguished by their phase velocity $v = \omega/k$, fast if $v > c$ and slow if $v < c$, and by the values of the derivatives of φ averaged over a period of the wave. In region I $\langle\partial\varphi/\partial t\rangle = \langle\partial\varphi/\partial x\rangle = 0$, while in region II only one of these values is different from zero. These regions are shown on the phase plane of fast steady-state waves in Fig. 1. Steady-state waves are expressed analytically in terms of elliptic functions. In region I we have

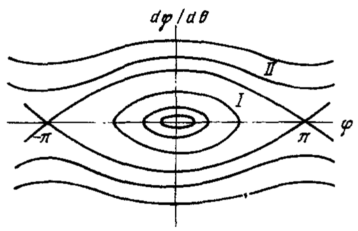

Fig. 1

$$\varphi = 2\arcsin\left\{s\,\mathrm{Sn}\left[\frac{2K(s)}{\pi}\theta;\ s\right]\right\} + \pi I(v^2 - c^2)$$
$$\omega^2 = c^2k^2 + \frac{c^2}{\Lambda^2}\left[\frac{\pi}{2K(s)}\right]^2 \mathrm{sign}\,(v^2 - c^2) \tag{1.2}$$

and in region II

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$$\varphi = 2\,\mathrm{am}\left[\frac{K(s)}{\pi}\theta,\ s\right] + \pi I(v^2 - c^2) \tag{1.3}$$

$$\omega^2 = c^2k^2 + \frac{c^2}{\Lambda^2}\left(\frac{\pi}{sK(s)}\right)^2 \mathrm{sign}\,(v^2 - c^2)$$

where $\theta = \omega t - kx$, I(z) is the Heaviside step function, am(z) and Sn(z) are, respectively, the amplitude and the Jacobian Sinus amplitudinis, and K(s) is the complete elliptic integral of the first kind of modulus s.

For a fixed k the shape of the slow waves, which is fixed by the parameter s, cannot be arbitrary. Only those values of s are possible for which $\omega^2 > 0$; there are no such restrictions for fast waves.

2. Taking account of dissipative factors in many cases leads to an addition to the Klein–Gordon equation of a term of the type $\partial\varphi/\partial t$, and the initial equation takes the form

$$\frac{\partial^2\varphi}{\partial t^2} - c^2\frac{\partial^2\varphi}{\partial x^2} + \frac{c^2}{\Lambda^2}\sin\varphi + \frac{1}{\tau}\frac{\partial\varphi}{\partial t} = 0 \tag{2.1}$$

For large enough τ it can be assumed that the solution of Eq. (2.1) is given by (1.2) or (1.3), where the amplitude (or modulus s) and the frequency are variable. Since the envelopes vary slowly it is convenient to obtain their equation by using the method of averaging, for example, of a generalized variational principle [15, 16]. Lagrangian and Rayleigh functions can be found for Eq. (2.1). After averaging them over the phase θ we obtain the required equations

$$\frac{\partial}{\partial t}\left[\omega\langle\varphi_\theta^2\rangle\right] + c^2\frac{\partial}{\partial x}\left[k\langle\varphi_\theta^2\rangle\right] = -\frac{\omega}{\tau}\langle\varphi_\theta^2\rangle \tag{2.2}$$

$$\partial k/\partial t + \partial\omega/\partial x = 0$$

We set up an initial-value problem for (2.2); i.e., we assume that at $t=0$ the solution has the form (1.2) or (1.3) with given values of s_0 and k_0. Then for $t>0$ the wave number is not changed but s will vary with time. Under these conditions (2.2) can be integrated:

$$I(s, k_0\Lambda)\exp(t/\tau) = \mathrm{const} \tag{2.3}$$

where in region I the function I is

$$I = \left[K^2(s) + \frac{\pi^2}{4k_0^2\Lambda^2}\mathrm{sign}\,(v^2 - c^2)\right]^{1/2}[E(s) - (1 - s^2)K(s)] \tag{2.4}$$

and in region II

$$I = \left[K^2(s) + \frac{\pi^2}{k_0^2\Lambda^2}s^{-2}\,\mathrm{sign}\,(v^2 - c^2)\right]^{1/2}E(s) \tag{2.5}$$

Here E(s) is the complete elliptic integral of the second kind of modulus s. Figure 2 shows I(s) for 1) region I, $v > c$; 2) region I, $v < c$; 3) region II, $v < c$; 4) region II, $v > c$. Equation (2.3) determines the dependence of s implicitly, and by using (1.2) or (1.3) the time dependence of the remaining elements of the wave. The basic features in the behavior of the solution are determined by I(s). In region I, I(s) is a monotonic function of its argument, and as a result s(t) decreases and the wave becomes more and more sinusoidal. If $v > c$ the amplitude of the wave decreases to zero, a process which is qualitatively similar to that described in [15]. With an increase in amplitude the damping decrement is increased and for $s = 1$ is twice as large as the value obtained in the linear theory. Thus, a fast wave of finite amplitude is damped more rapidly in region I than follows from the linear approximation.

When $k_0\Lambda < 1$ a new effect is possible for the slow wave in region I, related to the damping out of the wave over a finite distance but after an infinite time. In this case the frequency of the wave vanishes, and the process reaches a steady state $\varphi = \varphi(x)$. The dependence of the parameter s_1^* on $k_0\Lambda$, characterizing the limiting shape of the slow wave in region I as $t \to \infty$, is shown by curve 1 of Fig. 3.

For the slow wave in region II I(s) is also a monotonic function, and as $t \to \infty$ s approaches the asymptotic value $s_2^*(k_0\Lambda)$ (curve 2 of Fig. 3) for which $\omega = 0$. The wave is damped out in a finite distance also.

For the fast wave in region II I(s) has a minimum (Fig. 2) for a certain $s = s_3^*(k_0\Lambda)$ shown by curve 3 of Fig. 3. Then after a finite time, independently of the initial conditions, the shape parameter s(t) approaches s_3^*, after which Eq. (2.3) loses its meaning. The wave parameters cannot be smoothly "retuned," and the quasistationary wave structure is destroyed. The conditions arising in this case do not yield to analysis. If we neglect the nonlinearity by formally letting $\Lambda \to \infty$, the decreasing portion of the graph of I(s) vanishes, and I(s) becomes monotonic. Thus, the disruption of the steady-state structure of the wave is due entirely to the nonlinearity.

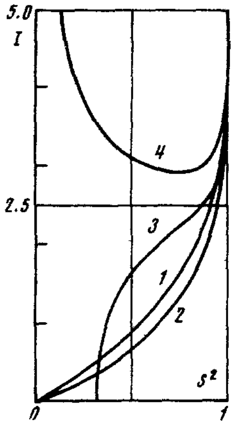

Fig. 2

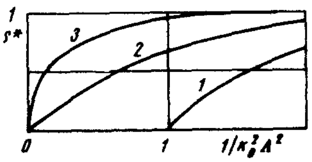

Fig. 3

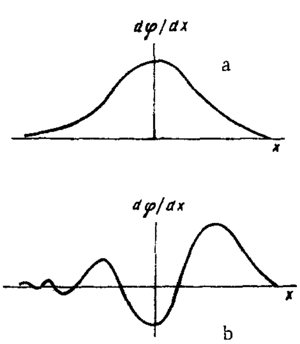

Fig. 4

3. We consider a boundary value problem when the wave has a steady-state shape at $x = 0$. In this case we find $s(x)$ from the equation

$$\frac{dR}{dx} = -\frac{\omega}{\tau c_0^2}\langle\varphi_\theta^2\rangle, \quad R = k\langle\varphi_\theta^2\rangle \tag{3.1}$$

where ω = const and k is determined by the dispersion relations. Although the solution of Eq. (3.1) is reduced to quadratures, its features are more conveniently analyzed by starting from the expressions for R and $\langle\varphi_\theta^2\rangle$ as functions of s. Since $\langle\varphi_\theta^2\rangle$ is positive, R decreases with distance to its minimum value. Since R(s) has a minimum for slow waves in region II, beginning with a certain distance Eq. (3.1) becomes contradictory, implying the impossibility of the propagation of a steady-state wave of this type to large distances. For fast waves in regions II and I (in the latter case for $\Lambda\omega < c$) R(s) is defined only for $s > s_0$ ($s_0 > 0$) and the dependence on s is monotonic. Since $\langle\varphi_\theta^2\rangle$ is different from zero at point s_0, the steady-state structure of the wave cannot exist at large distances from the boundary. In this case care must be used in speaking of the disruption of the wave, since close to the critical point $k \to 0$, and the method of averaging is generally inapplicable.

When $\Lambda\omega > c$ Eq. (3.1) is applicable for slow and fast waves at all distances and the damping of a steady-state wave can be completely described by means of it.

4. In addition to periodic waves of essentially nonsinusoidal shape ($s = 1$) solitons can be propagated which correspond to phase differences of 2π, and therefore are called 2π-pulses in nonlinear optics [17]. Their propagation can be described by using (2.2) in the limit $k \to 0$ and $s \to 1$ in such a way that kK(s) remains finite. This quantity determines the velocity of the soliton by the equation

$$kK(s) = \pi c / \Lambda\sqrt{|v^2 - c^2|} \tag{4.1}$$

By using (4.1) we obtain for I

$$I = [v^2 / |v^2 - c^2|]^{1/2} \tag{4.2}$$

Substituting (4.2) into (2.3) we find for v(t)

$$v(t) = c\,[1 - (1 - c^2/v_0^2)\exp 2t/\tau]^{-1/2} \tag{4.3}$$

where $v_0 = v(t=0)$. Hence, it follows that the velocity of the fast wave increases without bound in the finite time $\Delta = -\frac{1}{2}\ln(1 - c^2/v_0^2)$, and during this time the soliton travels the finite distance $L = \frac{1}{2}c\tau\ln[(c+v_0)/(v_0 - c)]$. As $t \to T$ $\partial\varphi/\partial x \to 0$ and $\partial\varphi/\partial t$ approaches a finite value. For $t > T$ the pulse cannot remain steady.

As $t \to \infty$, $\partial\varphi/\partial t \to 0$, and $\partial\varphi/\partial x$ for the slow wave approaches a constant value. Although the damping in this case lasts an infinitely long time, the soliton travels the finite distance $L = \frac{1}{2}c\tau\ln[(c+v_0)/(c - v_0)]$. Consequently, because of damping a single wave cannot propagate to large distances.

5. In addition to the 2π-pulses mentioned, π-pulses are also possible because of dissipation. These are steady-state waves and bring about a relaxation of the medium from an inverted state with the phase $\pm\pi$ to the normal state with $\varphi = 0$ (cf. [17]). It can be seen from (1.1) that the solution $\varphi = \pm\pi$ is unstable with respect to perturbations with a scale larger than Λ. In contrast with 2π-pulses a π-pulse solution in a dispersive medium can be found for any τ. Figure 4a shows the solution for small τ and Fig. 4b for large τ. In the $\tau \to 0$ approximation we have

$$\frac{d\varphi}{dx} = \frac{c^2\tau}{v\Lambda}\,\mathrm{sech}\,\frac{c^2\tau(x - vt)}{\Lambda^2 v} \tag{5.1}$$

This solution has the same form as that for a 2π-pulse. Solution (5.1) is exact for $v = c$ and any τ.

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